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# A note on Nordhaus–Gaddum inequalities for domination

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**Abstract**

For a graph $G$ of order $n$, let $\gamma(G)$, $\gamma_2(G)$ and $\gamma_t(G)$ be the domination, double domination and total domination numbers of $G$, respectively. The minimum degree of the vertices of $G$ is denoted by $\delta(G)$ and the maximum degree by $\Delta(G)$. In this note we prove a conjecture due to Harary and Haynes saying that if a graph $G$ has $\gamma(G), \gamma(\bar{G}) \geqslant 4$, then

$$\gamma_2(G) + \gamma_2(\bar{G}) \leqslant n - \Delta(G) + \delta(G) - 1 \leqslant n - 1$$

and

$$\gamma_t(G) + \gamma_t(\bar{G}) \leqslant n - \Delta(G) + \delta(G) - 1 \leqslant n - 1,$$

where $\bar{G}$ is the complement of $G$.


*Keywords:* Domination; Total domination; Double domination

## 1. Introduction

The graphs considered here are finite and simple. Let $G$, $\bar{G}$ denote a graph and its complement. The $V = V(G)$ is the vertex set of $G$ and $E = E(G)$ is the edge set of $G$. The order $|V(G)|$ of $G$ is denoted by $n$. The *neighborhood* of a vertex $v \in V$ is defined as $N(v) = \{u \in V | uv \in E\}$. The *close neighborhood* of a vertex $v$ is $N[v] = N(v) \cup \{v\}$. The *degree* of $v$ is $d(v) = |N(v)|$. The maximum degree of a graph $G$ is denoted by $\Delta(G)$

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and the minimum degree is denoted by $\delta(G)$. A vertex in $G$ is said to dominate itself and all vertices adjacent to it. A set $S \subseteq V$ is called a *dominating set* if each vertex in $V - S$ is adjacent to some vertex in $S$. The *domination number* $\gamma(G)$ of a graph $G$ is the minimum cardinality of a dominating set of $G$. If $S$ is a subset of $V$, we use $\langle S\rangle$ to denote the induced subgraph in $G$ by $S$. For $v \in V$, we set $d_S(v) = |\{u \in S \mid uv \in E\}|$.

Extensive studies on domination-related topics have been done in the last 30 years. In [4] Harary first introduced the concept of double dominating set. A set $S \subseteq V$ is called *a double dominating set* for $G$ if each vertex in $V$ is dominated by at least two vertices in $S$ . The *double domination number*, denoted by $\gamma_2(G)$, is the smallest size of a double dominating set. More general, Fink and Jacobson [1,2] introduced the concept of $k$-domination number. For a positive integer $k$, a set $S$ is called a *$k$-dominating set* if each vertex in $V$ is dominated by at least $k$ vertices in $S$. The *$k$-domination number*, denoted by $\gamma_k(G)$, is the minimum cardinality of a $k$-dominating set of $G$. *A total dominating set* of $G$ is a subset $S$ of $V$ such that each vertex in $V$ is adjacent to a vertex of $S$. *The total domination number*, *denoted by* $\gamma_t(G)$, is the minimum cardinality of a total dominating set.

Let $u(G)$ be a graph parameter, the calculations of extremum values of $u(G)+u(\bar{G})$ and $u(G)u(\bar{G})$ taken over all $n$-vertex graphs $G$ are known as Nordhaus–Gaddum type problems, due to the results of [5]. In this note we prove a conjecture about Nordhous–Gaddum inequalities for domination.

## 2. Main results

In [3] Harary and Haynes proved the following result and proposed the following conjecture.

**Theorem 1** (Harary and Haynes [3]). *If $G$ is a graph with $\gamma(G), \gamma(\bar{G}) \geqslant 5$, then*

$$\gamma_2(G) + \gamma_2(\bar{G}) \leqslant n - \Delta(G) + \delta(G) - 1.$$

**Conjecture** (Harary and Haynes [3]). *If $G$ is a graph with $\gamma(G), \gamma(\bar{G}) \geqslant 4$, then*

(1) $\gamma_2(G) + \gamma_2(\bar{G}) \leqslant n - \Delta(G) + \delta(G) - 1 \leqslant n - 1$,
(2) $\gamma_t(G) + \gamma_t(\bar{G}) \leqslant n - \Delta(G) + \delta(G) - 1$.

Our main result below strengths the conjecture.

**Theorem 2.** *For any integer $k \geqslant 1$, if a graph $G$ has $\gamma(G), \gamma(\bar{G}) \geqslant k+2$, then*

$$\gamma_k(G) + \gamma_k(\bar{G}) \leqslant n - \Delta(G) + \delta(G) - 1.$$

**Proof.** Given $G$ with $\gamma(G), \gamma(\bar{G}) \geqslant k+2$, consider an arbitrary pair of vertices $x$ and $y$ in $\bar{G}$. There exists a set of vertices $W \subseteq V$ such that $W$ is not dominated by $\{x, y\}$ in $\bar{G}$ and $(V - W)$ is dominated by $\{x, y\}$ in $\bar{G}$. Let $W_1$ be a maximum sized independent set for $\langle W\rangle$ in $\bar{G}$, then $|W_1| \geqslant \gamma(\bar{G}) - 2 \geqslant k$. Otherwise $\gamma(\bar{G}) \leqslant |W_1| + 2 < k+2$, a contradiction. Now $W_1, W_1 \cup \{x\}, W_1 \cup \{y\}$ are independent sets in $\bar{G}$. Thus, in $G$ each pair of vertices must have at least $\gamma(\bar{G}) - 2 \geqslant k$ common neighbors. Furthermore, these neighbors are mutually adjacent in $G$.

Let $v$ be a vertex of minimal degree of $G$, and $S = \{u \in V(G) \mid uv \in E(G)\}$, then $S$ is a vertex cutset of $G$. Let vertices $x$ and $y$ be in separate components of $G - S$. Hence, $x$ and $y$ have at least $\gamma(\bar{G}) - 2 \geqslant k$ mutually adjacent common neighbors and these neighbors must be in $S$. Then every vertex in $G - S$ is dominated by at least $k$ vertices in $S$.

Next, we show that every vertex in $S$ is dominated by at least $k$ vertices in $S$. Assume $v_1$ is a vertex in $S$ with $d_S(v_1) < k - 1$, then there are at most $k - 2$ common neighbors for $v$ and $v_1$, this is a contradiction. So $S$ is a $k$-dominating set of $G$ with size $\delta(G)$, thus we have $\gamma_k(G) \leqslant \delta(G)$. Similarly, $\gamma_k(\bar{G}) \leqslant \delta(\bar{G}) = n - 1 - \Delta(G)$. Then $\gamma_k(G) + \gamma_k(\bar{G}) \leqslant n - \Delta(G) + \delta(G) - 1 \leqslant n - 1$. This completes the proof. $\square$

The next result is immediate from Theorem 2.

**Corollary 3.** *If a graph $G$ has $\gamma(G), \gamma(\bar{G}) \geqslant 4$, then*

$$\gamma_2(G) + \gamma_2(\bar{G}) \leqslant n - \Delta(G) + \delta(G) - 1 \leqslant n - 1.$$

**Theorem 4.** *If a graph $G$ has $\gamma(G), \gamma(\bar{G}) \geqslant 3$, then*

$$\gamma_t(G) + \gamma_t(\bar{G}) \leqslant n - \Delta(G) + \delta(G) - 1.$$

**Proof.** Given $G$ with $\gamma(G), \gamma(\bar{G}) \geqslant 3$, choose a vertex $v$ of $G$ such that $d(v) = \delta(G)$. Suppose $x$ is another vertex of $G$, and let $S$ be the set of vertices adjacent to both $v$ and $x$ in $G$. For any vertex $w$ of $V(\bar{G}) - \{v, x\}$, if $vw \notin E(\bar{G})$ and $xw \notin E(\bar{G}\}$, then $w \in S$. So $\{v, x\} \cup S$ is a dominating set of $\bar{G}$. This implies $|\{v, x\} \cup S| \geqslant \gamma(\bar{G}) \geqslant 3$; and so $|S| \geqslant 1$. Thus, every vertex in $V(G) - \{v\}$ is adjacent to some vertex in $N(v)$. Therefore, $N(v)$ is a total dominating set of $G$, and we have $\gamma_t(G) \leqslant |N(v)| = d(v) = \delta(G)$. Similarly, $\gamma_t(\bar{G}) \leqslant \delta(\bar{G}) = n - 1 - \Delta(G)$. The result follows. $\square$

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